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A Lorentz covariant holoraumy-induced "gadget" from minimal off-shell 4D, $\mathcal{N} = 1$ supermultiplets

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Abstract: Starting from three minimal off-shell 4D, $\mathcal{N} = 1$ supermultiplets, using constructions solely defined within the confines of the four dimensional field theory we show the existence of a "gadget" — a member of a class of metrics on the representation space of the supermultiplets — whose values directly and completely correspond to the values of a metric defined on the 1d, $N = 4$ adinkra networks adjacency matrices corresponding to the projections of the four dimensional supermultiplets.

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Contents

1 Introduction

The main purpose of this work is to extend a mathematical gadget previously found to exist in the realm of supersymmetrical quantum mechanics models [1–4] into four dimensional field theory with simple supersymmetry. This will provide a new example of "SUSY Holography" [5] — SUSY QM can realize aspects of SUSY QFT.

As we shall show in section two, it is possible (within purely four dimensional supersymmetrical field theories) to uncover the existence of a "Lorentz covariant fermionic holoraumy tensor" similar to that discovered within one dimensional models [1–4] *but with no reference whatsoever to lower dimensional constructs.* In turn this permits the definition of a mathematical gadget that relates to the properties of adjacency matrices [6, 7] of bipartite graphs (given the name '*adinkras*') [8]–[15]. This is similar to the results of [16], wherein it was shown that a certain parameter (χ_o – initially discovered in the study of four-color adinkra networks) can be defined solely using concepts from 4D, $\mathcal{N} = \infty$ superfield theory constructions.

We will next review the construction of the gadget for 1d supermultiplets based on adinkra networks in the following section. This discussion in the past has been shown to lead to a metric on the representation space of adinkra networks.

We follow this with a discussion containing a plausibility argument for why the commutator (as opposed to the anti-commutator) of supercharges provides an appropriate starting point in discussions of a representation space metric.

We conclude with a summary and observations and include one appendix with details useful for some of the calculations.

2 A 4D, $\mathcal{N} = 1$ minimal supermultiplet gadget

In this section, we wish to perform some calculations solely within the context of off-shell 4D, $\mathcal{N} = 1$ minimal supermultiplets. As is well known, there are essentially three such

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supermultiplets;[1] the chiral supermultiplet (CS), vector supermultiplet (VS), and tensor supermultiplet (TS).

Each such supermultiplet contains 4 bosonic degrees of freedom and 4 fermionic degrees of freedom. In reaching this conclusion, the degrees of freedom of gauge fields are counted only using their gauge-independent ones. The supermultiplets are specified by a set of component fields and the action of the D_a operators as realized on the component fields according to the following rules.

Chiral Supermultiplet: $(\boldsymbol{A},\, \boldsymbol{B},\, \boldsymbol{\psi_a},\, \boldsymbol{F},\, \boldsymbol{G})$

$$\begin{aligned} &\mathrm{D}_a A = \psi_a\,, \qquad\qquad \mathrm{D}_a B = i\,(\gamma^5)_a{}^b\,\psi_b\,,\\ &\mathrm{D}_a \psi_b = i\,(\gamma^\mu)_{a\,b}\,\partial_\mu A - (\gamma^5\gamma^\mu)_{a\,b}\,\partial_\mu B - i\,C_{a\,b}\,F + (\gamma^5)_{a\,b} G\,,\\ &\mathrm{D}_a F = (\gamma^\mu)_a{}^b\,\partial_\mu\,\psi_b\,, \qquad \mathrm{D}_a G = i\,(\gamma^5\gamma^\mu)_a{}^b\,\partial_\mu\,\psi_b\,, \end{aligned} \tag{2.1}$$

Vector Supermultiplet: $(\boldsymbol{A_\mu},\, \boldsymbol{\lambda_b},\, \mathbf{d})$

$$\begin{aligned} &\mathrm{D}_a\, A_\mu = (\gamma_\mu)_a{}^b\,\lambda_b\,,\\ &\mathrm{D}_a \lambda_b = -\,i\,\frac{1}{4}([\,\gamma^\mu\,,\,\gamma^\nu\,])_{a\,b}\,(\,\partial_\mu\,A_\nu - \partial_\nu\,A_\mu\,) + (\gamma^5)_{a\,b}\,\mathrm{d}\,,\\ &\mathrm{D}_a\,\mathrm{d} = i\,(\gamma^5\gamma^\mu)_a{}^b\;\partial_\mu \lambda_b\,, \end{aligned} \tag{2.2}$$

Tensor Supermultiplet: $(\boldsymbol{\varphi},\, \boldsymbol{B_{\mu\,\nu}},\, \boldsymbol{\chi_a})$

$$\begin{aligned} &\mathrm{D}_a \varphi = \chi_a\,, \qquad \mathrm{D}_a B_{\mu\,\nu} = -\,\frac{1}{4}([\,\gamma_\mu\,,\,\gamma_\nu\,])_a{}^b\,\chi_b\,,\\ &\mathrm{D}_a \chi_b = i\,(\gamma^\mu)_{a\,b}\,\partial_\mu \varphi - (\gamma^5\gamma^\mu)_{a\,b}\,\epsilon_\mu{}^{\rho\,\sigma\,\tau}\partial_\rho B_{\sigma\,\tau}\,. \end{aligned} \tag{2.3}$$

Up to gauge transformations, these satisfy the equation

$$\{\mathrm{D}_a\,, \mathrm{D}_b\} = i\,2\,(\gamma^\mu)_{a\,b}\,\partial_\mu\;, \tag{2.4}$$

when calculated upon each component field.

However, it is also possible to use the results in equations (2.1), (2.2), and (2.3) to calculate instead the commutators of the D_a operators as evaluated on each field. We find the following results which extend those found in [3].

Chiral Supermultiplet

$$\begin{aligned} &[\,\mathrm{D}_a\,,\ \mathrm{D}_b\,]A = -i\,2\,C_{ab}F + 2(\gamma^5)_{ab}G - 2(\gamma^5\gamma^\mu)_{ab}\partial_\mu B\,,\\ &[\,\mathrm{D}_a\,,\ \mathrm{D}_b\,]B = i\,2\,C_{ab}G + 2(\gamma^5)_{ab}F + 2(\gamma^5\gamma^\mu)_{ab}\partial_\mu A\,,\\ &[\,\mathrm{D}_a\,,\ \mathrm{D}_b\,]\psi_c = -i(\gamma^5\gamma^\mu)_{ab}(\gamma^5[\gamma_\mu\,,\,\gamma^\sigma])_c{}^d\partial_\sigma\psi_d\,,\\ &[\,\mathrm{D}_a\,,\ \mathrm{D}_b\,]F = -i\,2\,C_{ab}\eta^{\mu\sigma}\partial_\mu\partial_\sigma A + 2(\gamma^5)_{ab}\eta^{\mu\sigma}\partial_\mu\partial_\sigma B - 2(\gamma^5\gamma^\mu)_{ab}\partial_\mu G\,,\\ &[\,\mathrm{D}_a\,,\ \mathrm{D}_b\,]G = i\,2\,C_{ab}\eta^{\mu\sigma}\partial_\mu\partial_\sigma B + 2(\gamma^5)_{ab}\eta^{\mu\sigma}\partial_\mu\partial_\sigma A + 2(\gamma^5\gamma^\mu)_{ab}\partial_\mu F\,, \end{aligned} \tag{2.5}$$

[1] For the purposes of our discussion, we will here ignore the existence of variant representations and parity reflected representations of these three basic ones.

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Vector Supermultiplet

$$\begin{aligned}[][\mathrm{D}_a\,,\;\mathrm{D}_b\,]A_\mu &= -2\epsilon^{\sigma\nu}{}_{\mu\alpha}(\gamma^5\gamma^\alpha)_{ab}\partial_\sigma A_\nu - 2(\gamma^5\gamma_\mu)_{ab}\mathrm{d}\,,\\ [\mathrm{D}_a\,,\;\mathrm{D}_b\,]\mathrm{d} &= 2(\gamma^5\gamma^\mu)_{ab}\partial^\nu\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\,,\\ [\mathrm{D}_a\,,\;\mathrm{D}_b\,]\lambda_c &= -i\,2\,C_{ab}(\gamma^\mu)_c{}^d\partial_\mu\lambda_d - i\,2\,(\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d\partial_\mu\lambda_d\\ &\quad - i\,2\,(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d\partial_\mu\lambda_d\,,\end{aligned}\tag{2.6}$$

Tensor Supermultiplet

$$\begin{aligned}[][\mathrm{D}_a\,,\;\mathrm{D}_b\,]\varphi &= 2(\gamma^5\gamma^\mu)_{ab}\epsilon^\rho{}_\mu{}^{\alpha\beta}\partial_\rho B_{\alpha\beta}\,,\\ [\mathrm{D}_a\,,\;\mathrm{D}_b\,]B_{\mu\nu} &= -\epsilon_{\mu\nu}{}^\alpha{}_\beta(\gamma^5\gamma^\beta)_{ab}\partial_\alpha\varphi + 4(\gamma^5\gamma_{[\nu})_{ab}\epsilon^\rho{}_{\mu]}{}^{\alpha\beta}\partial_\rho B_{\alpha\beta}\,,\\ [\mathrm{D}_a\,,\;\mathrm{D}_b\,]\chi_c &= i\,2\,C_{ab}(\gamma^\mu)_c{}^d\partial_\mu\chi_d - i\,2\,(\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d\partial_\mu\chi_d\\ &\quad + i\,2\,(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d\partial_\mu\chi_d\,.\end{aligned}\tag{2.7}$$

Let us further focus only on the results for the fermions in each of the calculations in (2.5), (2.6), and (2.7). This brings us to the results below.

Chiral Supermultiplet Fermion

$$\begin{aligned}[][\mathrm{D}_a\,,\;\mathrm{D}_b\,]\,\psi_c &= -i\,(\gamma^5\gamma^\nu)_{ab}(\gamma^5[\,\gamma_\nu\,,\;\gamma^\mu\,])_c{}^d\partial_\mu\psi_d\\ &\equiv \left[\boldsymbol{H}^{\mu(CS)}\right]_{a\,b\,c}{}^d\,\left(\partial_\mu\psi_d\,\right)\,,\end{aligned}\tag{2.8}$$

Vector Supermultiplet Fermion

$$\begin{aligned}[][\mathrm{D}_a\,,\;\mathrm{D}_b\,]\,\lambda_c &= -i2\,C_{ab}(\gamma^\mu)_c{}^d\partial_\mu\lambda_d - i2\,(\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d\partial_\mu\lambda_d\\ &\quad - i2\,(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d\partial_\mu\lambda_d\\ &\equiv \left[\boldsymbol{H}^{\mu(VS)}\right]_{a\,b\,c}{}^d\,\left(\partial_\mu\lambda_d\,\right)\,,\end{aligned}\tag{2.9}$$

Tensor Supermultiplet Fermion

$$\begin{aligned}[][\mathrm{D}_a\,,\;\mathrm{D}_b\,]\,\chi_c &= i2\,C_{ab}(\gamma^\mu)_c{}^d\partial_\mu\chi_d - i2\,(\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d\partial_\mu\chi_d\\ &\quad + i2\,(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d\partial_\mu\chi_d\\ &\equiv \left[\boldsymbol{H}^{\mu(TS)}\right]_{a\,b\,c}{}^d\,\left(\partial_\mu\chi_d\,\right)\,.\end{aligned}\tag{2.10}$$

We derive (directly from the four dimensional formulations of each of (CS), (VS), and (TS) cases respectively) associated quantities $\left[\boldsymbol{H}^{\mu(CS)}\right]_{a\,b\,c}{}^d$, $\left[\boldsymbol{H}^{\mu(VS)}\right]_{a\,b\,c}{}^d$, and $\left[\boldsymbol{H}^{\mu(TS)}\right]_{a\,b\,c}{}^d$. These are holoraumy tensors defined purely in terms of four dimensional Lorentz covariant concepts. Unlike their SUSY QM analogues [2–4], these also carry a Lorentz vector index. Since each of these 4D holoraumy tensors carries additional four spinor-indices, by performing contractions over these spinor indices we can also form mathematical gadgets (i.e. — proposed metrics on the representation spaces) similar to those discussed in the one-dimensional cases.

Guided by our experience from working with 0-brane reductions, we require a Lorentz covariant gadget (denoted by $\widehat{\mathcal{G}}[(\widehat{\mathcal{R}}),(\widehat{\mathcal{R}}')\,])$ defined over the four dimensional supermultiplet representations $(\widehat{\mathcal{R}})$ and $(\widehat{\mathcal{R}}')$. We propose an ansatz of the form

$$\begin{aligned}\widehat{\mathcal{G}}[(\widehat{\mathcal{R}}),(\widehat{\mathcal{R}}')] = {} & m_1\,[\boldsymbol{H}^{\mu(\widehat{\mathcal{R}})}]_{a\,b\,c}{}^{d}\,[\boldsymbol{H}_{\mu}{}^{(\widehat{\mathcal{R}}')}]^{a\,b}{}_{d}{}^{c} \\ & + m_2\,(\gamma^{\alpha})_{c}{}^{e}\,[\boldsymbol{H}^{\mu(\widehat{\mathcal{R}})}]_{a\,b\,e}{}^{f}\,(\gamma_{\alpha})_{f}{}^{d}\,[\boldsymbol{H}_{\mu}{}^{(\widehat{\mathcal{R}}')}]^{a\,b}{}_{d}{}^{c} \\ & + m_3\,([\,\gamma^{\alpha}\,,\ \gamma^{\beta}\,])_{c}{}^{e}\,[\boldsymbol{H}^{\mu(\widehat{\mathcal{R}})}]_{a\,b\,e}{}^{f}\,([\,\gamma_{\alpha}\,,\ \gamma_{\beta}\,])_{f}{}^{d}\,[\boldsymbol{H}_{\mu}{}^{(\widehat{\mathcal{R}}')}]^{a\,b}{}_{d}{}^{c} \\ & + m_4\,(\gamma^{5}\gamma^{\alpha})_{c}{}^{e}\,[\boldsymbol{H}^{\mu(\widehat{\mathcal{R}})}]_{a\,b\,e}{}^{f}\,(\gamma^{5}\gamma_{\alpha})_{f}{}^{d}\,[\boldsymbol{H}_{\mu}{}^{(\widehat{\mathcal{R}}')}]^{a\,b}{}_{d}{}^{c} \\ & + m_5\,(\gamma^{5})_{c}{}^{e}\,[\boldsymbol{H}^{\mu(\widehat{\mathcal{R}})}]_{a\,b\,e}{}^{f}\,(\gamma^{5})_{f}{}^{d}\,[\boldsymbol{H}_{\mu}{}^{(\widehat{\mathcal{R}}')}]^{a\,b}{}_{d}{}^{c}\,,\end{aligned} \tag{2.11}$$

and then seek to find constants m_1, m_2, m_3, m_4, and m_5, such that the following equation for the Lorentz covariant scalar $\widehat{\mathcal{G}}[(\widehat{\mathcal{R}}),(\widehat{\mathcal{R}}')]$ takes the explicit form

$$\widehat{\mathcal{G}}[(\widehat{\mathcal{R}}),(\widehat{\mathcal{R}}')] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}. \tag{2.12}$$

to agree with adinka-based results in [1, 2]. There exists an infinite number of such solutions. One solution is given by $m_1 = -\frac{1}{768}$, $m_2 = m_4 = \frac{1}{1{,}536}$, and $m_3 = m_5 = 0$, but as long as the following three equations are satisfied

$$\begin{aligned} m_1 + 16\,m_3 + m_5 &= -\frac{1}{768}\,, \\ m_1 - 48\,m_3 + 8\,m_4 + m_5 &= \frac{3}{768}\,, \\ m_1 + 4\,m_2 + 48\,m_3 - 4\,m_4 - 3\,m_5 &= -\frac{1}{768}\,, \end{aligned} \tag{2.13}$$

the result in (2.12) will be valid.

3 The gadget in Valise-Adinkra networks

To readers who have been following our explorations for some time [1–4], the matrix denoted by $\widehat{\mathcal{G}}[\,(\widehat{\mathcal{R}}),(\widehat{\mathcal{R}}')\,]$ in (2.12) should be very familiar. However, the initial appearance of this matrix (in the first of these references) was derived by methods that begin with valise adinkra networks. It is useful to recount that derivation.

In the works of [8]–[15], among others, it has been argued there exist networks which can be drawn in the forms of graphs that encode the representation theory of off-shell supersymmetrical multiplets in higher dimensional theories. All such graphs to date have been constructed from a starting point of ones with cubical topology, but in order to irreducibly describe supersymmetry representations the nodes in such cubical graphs must be identified in a manner that uses error-correcting codes [9–11]. When the nodes of such graphs only appear at two levels, the graph is called a valise. Three example (the so-called (C)-chiral, (V)-vector, and (T)-tensor) valise adinkras are shown below.

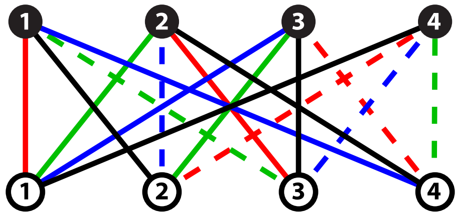

Figure 1. Illustration of a (C)hiral Valise Adinkra Network.

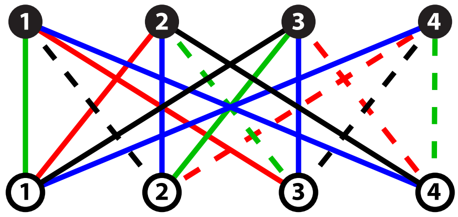

Figure 2. Illustration of a (V)ector Valise Adinkra Network.

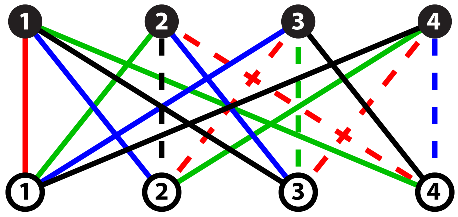

Figure 3. Illustration of a (T)ensor Valise Adinkra Network.

A standard concept in graph theory is that a network possesses an "adjacency matrix." In our work, we have taken this concept a step further. The links in adinkras fall into equivalence classes. Different classes are denoted by distinct colors in the diagrams. The distinct heights in the graphs denote distinct engineering dimensions associated with the nodes. Finally, links appeared dashed or not to indicate the presences of minus signs or not. Thus, we replace the traditional adjacency graphs by more elaborate "L-matrices" and "R-matrices." Alternately, if all colored links are replaced by black lines and all dashing is dropped we recover a standard adjacency matrix.

For minimal representations, the R-matrices correspond to the matrix transposed version of the L-matrices. For the respective three illustrations shown above, we have the following three respective sets of L-matrices shown below. From here on we use a representation label $\mathcal{R}$ (without the 'hat') to denoted different adrinkra network representations. This is in distinction with the representation label $\widehat{\mathcal{R}}$ used for different supermultiplets in the last section. From the work in [1, 2] we have

(C)

$$
(\mathrm{L}_1)_{i\hat{k}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad (\mathrm{L}_2)_{i\hat{k}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},
$$

$$
(\mathrm{L}_3)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad (\mathrm{L}_4)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{3.1}
$$

(V) ___

$$
(\mathrm{L}_1)_{i\hat{k}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad (\mathrm{L}_2)_{i\hat{k}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},
$$

$$
(\mathrm{L}_3)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad (\mathrm{L}_4)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{3.2}
$$

(T) ___

$$
(\mathrm{L}_1)_{i\hat{k}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad (\mathrm{L}_2)_{i\hat{k}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(\mathrm{L}_3)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad (\mathrm{L}_4)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{3.3}
$$

and given the L-matrices associated with any of the adinkras, we can define another set of matrices (the $\widetilde{V}$ matrices [2]) via the equations.

$$
(\widetilde{V}^{(\mathcal{R})}_{\mathrm{IJ}})_{\hat{i}}{}^{\hat{k}} = \frac{1}{2}\left[(\,\mathrm{R}^{(\mathcal{R})}_{\mathrm{I}}\,)_{\hat{i}}{}^{j}\,(\,\mathrm{L}^{(\mathcal{R})}_{\mathrm{J}}\,)_{j}{}^{\hat{k}} - (\,\mathrm{R}^{(\mathcal{R})}_{\mathrm{J}}\,)_{\hat{i}}{}^{j}\,(\,\mathrm{L}^{(\mathcal{R})}_{\mathrm{I}}\,)_{j}{}^{\hat{k}} \right]. \tag{3.4}
$$

Since the L-matrices are dependent on which adinkra representation is taken as a starting point, this is also the case for the $\widetilde{V}$ matrices and we indicate this by including a representation label $\mathcal{R}$. Explicit calculations for each set leads to [1, 2]

$$
\begin{aligned}
(\widetilde{V}^{(\mathcal{R})}_{\mathrm{IJ}})_{\hat{i}}{}^{\hat{k}} = i &\left[\ell^{(\mathcal{R})1}_{\mathrm{IJ}}\,(\alpha^1)_{\hat{i}}{}^{\hat{k}} + \ell^{(\mathcal{R})2}_{\mathrm{IJ}}\,(\alpha^2)_{\hat{i}}{}^{\hat{k}} + \ell^{(\mathcal{R})3}_{\mathrm{IJ}}\,(\alpha^3)_{\hat{i}}{}^{\hat{k}} \right] \\
&+ i \left[\widetilde{\ell}^{(\mathcal{R})1}_{\mathrm{IJ}}\,(\beta^1)_{\hat{i}}{}^{\hat{k}} + \widetilde{\ell}^{(\mathcal{R})2}_{\mathrm{IJ}}\,(\beta^2)_{\hat{i}}{}^{\hat{k}} + \widetilde{\ell}^{(\mathcal{R})3}_{\mathrm{IJ}}\,(\beta^3)_{\hat{i}}{}^{\hat{k}} \right].
\end{aligned} \tag{3.5}
$$

with the non-vanishing entries for each representation taking the forms

$$
\begin{array}{llllll}
\ell^{(C)1}_{12} = 1 & \ell^{(C)2}_{13} = 1 & \ell^{(C)3}_{14} = 1 & \ell^{(C)1}_{23} = 1 & \ell^{(C)2}_{24} = -1 & \ell^{(C)3}_{34} = 1\,, \\
\widetilde{\ell}^{(V)1}_{12} = -1 & \widetilde{\ell}^{(V)2}_{13} = 1 & \widetilde{\ell}^{(VM)3}_{14} = -1 & \widetilde{\ell}^{(V)1}_{23} = 1 & \widetilde{\ell}^{(V)2}_{24} = 1 & \widetilde{\ell}^{(V)3}_{34} = 1\,, \\
\widetilde{\ell}^{(T)1}_{12} = 1 & \widetilde{\ell}^{(T)2}_{13} = 1 & \widetilde{\ell}^{(T)3}_{14} = 1 & \widetilde{\ell}^{(T)1}_{23} = -1 & \widetilde{\ell}^{(T)2}_{24} = 1 & \widetilde{\ell}^{(T)3}_{34} = -1\,.
\end{array} \tag{3.6}
$$

The 4×4 matrices $\boldsymbol{\alpha}^1$, $\boldsymbol{\alpha}^2$, $\boldsymbol{\alpha}^3$, $\boldsymbol{\beta}^1$, $\boldsymbol{\beta}^2$, $\boldsymbol{\beta}^3$ which appear in (3.5) can be written as

$$\begin{aligned} \boldsymbol{\alpha}^1 &= \boldsymbol{\sigma}^2 \otimes \boldsymbol{\sigma}^1 \,, & \boldsymbol{\alpha}^2 &= \mathbf{I}_2 \otimes \boldsymbol{\sigma}^2 \,, & \boldsymbol{\alpha}^3 &= \boldsymbol{\sigma}^2 \otimes \boldsymbol{\sigma}^3 \,, \\ \boldsymbol{\beta}^1 &= \boldsymbol{\sigma}^1 \otimes \boldsymbol{\sigma}^2 \,, & \boldsymbol{\beta}^2 &= \boldsymbol{\sigma}^2 \otimes \mathbf{I}_2 \,, & \boldsymbol{\beta}^3 &= \boldsymbol{\sigma}^3 \otimes \boldsymbol{\sigma}^2 \,. \end{aligned} \tag{3.7}$$

where the outer product conventions are in [12]. These matrices satisfy the identities

$$\begin{aligned} \boldsymbol{\alpha}^{\widehat{I}} \boldsymbol{\alpha}^{\widehat{K}} &= \delta^{\widehat{I}\widehat{K}} \mathbf{I}_4 + i\,\epsilon^{\widehat{I}\widehat{K}\widehat{L}} \boldsymbol{\alpha}^{\widehat{K}} \,, & \boldsymbol{\beta}^{\widehat{I}} \boldsymbol{\beta}^{\widehat{K}} &= \delta^{\widehat{I}\widehat{K}} \mathbf{I}_4 + i\,\epsilon^{\widehat{I}\widehat{K}\widehat{L}} \boldsymbol{\beta}^{\widehat{K}} \,, \\ \mathrm{Tr}\big(\boldsymbol{\alpha}^{\widehat{I}} \boldsymbol{\alpha}^{\widehat{J}}\big) = \mathrm{Tr}\big(\boldsymbol{\beta}^{\widehat{I}} \boldsymbol{\beta}^{\widehat{J}}\big) &= 4\,\delta^{\widehat{I}\widehat{J}} \,, & \mathrm{Tr}\big(\boldsymbol{\alpha}^{\widehat{I}} \boldsymbol{\beta}^{\widehat{J}}\big) &= 0 \,, \\ \mathrm{Tr}\big(\boldsymbol{\alpha}^{\widehat{I}}\big) = \mathrm{Tr}\big(\boldsymbol{\beta}^{\widehat{I}}\big) &= 0 \,. \end{aligned} \tag{3.8}$$

Finally, a gadget for the valise adinkra network can be defined by

$$\mathcal{G}\left[(\mathcal{R}),\,(\mathcal{R}')\right] = -\frac{1}{48}\sum_{\mathrm{I,J}} \mathrm{Tr}\left[\widetilde{V}^{(\mathcal{R})}_{\mathrm{I\,J}} \widetilde{V}^{(\mathcal{R}')}_{\mathrm{I\,J}}\right] = \frac{1}{12}\sum_{\mathrm{I,J},\hat{a}} \left[\ell^{(\mathcal{R})\hat{a}}_{\mathrm{IJ}}\,\ell^{(\mathcal{R}')\hat{a}}_{\mathrm{IJ}} + \widetilde{\ell}^{(\mathcal{R})\hat{a}}_{\mathrm{IJ}}\,\widetilde{\ell}^{(\mathcal{R}')\hat{a}}_{\mathrm{IJ}}\right], \tag{3.9}$$

and upon using the information in (3.5), we find

$$\mathcal{G}\left[(\mathcal{R}),\,(\mathcal{R}')\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} . \tag{3.10}$$

4 Why the commutator of supercharges?

In an epochal paper, D. Gross and R. Jackiw [17] noted a particular mathematical quantity in the representation theory of Lie Algebras, plays a prominent role with regard to anomalies in gauge theories. The quantity in question can be called the "d-coefficients tensor" (following conventions that arise in the context of the su(3) Lie algebra). For the purposes of our discussion we will write this in the form as

$$d^{(\mathcal{R})}_{A\,B\,C} = \frac{1}{2}\,\mathrm{Tr}\left[\left\{\boldsymbol{t}^{(\mathcal{R})}_A \,,\, \boldsymbol{t}^{(\mathcal{R})}_B\right\}\boldsymbol{t}^{(\mathcal{R})}_C\right] , \tag{4.1}$$

where the notation is indicative of several relevant features. In this expression there appear some matrices $\boldsymbol{t}^{(\mathcal{R})}_A$ (with $A = 1, 2, \ldots\, p$ for some integer p) in a representation $\mathcal{R}$ of some Lie algebra. This way of defining the d-coefficients has the advantage that for any set of matrices $\boldsymbol{t}^{(\mathcal{R})}_A$, this provides a a well-defined way to explicitly calculate them. We may let the symbol $d_{(\mathcal{R})}$ denote the dimension of $(\mathcal{R})$ and $d_{(\mathcal{R}')}$ denote the dimension of $(\mathcal{R}')$. Another result that follows from this definition, can be seen from the following consideration.

If the matrices $\boldsymbol{t}^{(\mathcal{R}')}_A$ of one representation (denoted by $(\mathcal{R}')$) are related to the matrices $\boldsymbol{t}^{(\mathcal{R})}_A$ via equations of the form

$$\boldsymbol{t}^{(\mathcal{R}')}_A = \mathcal{S}^{-1}\,\boldsymbol{t}^{(\mathcal{R})}_A \mathcal{S} \,, \tag{4.2}$$

for some matrix $\mathcal{S}$ and its inverse, then

$$d^{(\mathcal{R}')}_{A\,B\,C} = d^{(\mathcal{R})}_{A\,B\,C} \,. \tag{4.3}$$

However, the "d-coefficients tensor" can be calculated for any representation and the representation $(\mathcal{R}')$ need not be restricted to satisfy (4.2).

Clearly the values of these coefficients depend on the choice of how one orders the $\boldsymbol{t}$-matrices and to utilize a quantity that does not depend on the ordering we define a mathematical "gadget" (denoted by $\tilde{g}(\mathcal{R},\,\mathcal{R}'))$ on these representation spaces via the equation

$$\tilde{g}(\mathcal{R},\,\mathcal{R}') = \mathcal{N}_0 \sum_{A,\,B,\,C} d^{(\mathcal{R})\,C}_{A\,B}\, d^{(\mathcal{R}')\,C}_{A\,B}\,, \tag{4.4}$$

where $\mathcal{N}_0$ is a normalization constant whose value is fixed by requiring that $\tilde{g}(\mathcal{R},\,\mathcal{R}) = 1$ when $(\mathcal{R})$ denotes a minimal irreducible representation. If the d-coefficients defined in (4.1) are real, the gadget assigns a real number to the pair of representations $(\mathcal{R})$ and $(\mathcal{R}')$. Furthermore, whenever $(\mathcal{R}) = (\mathcal{R}')$, the gadget assigns a non-negative number if the d-coefficients are real.

A simple example of this formalism can be seen in the case where the Lie algebra is su(3), with $(\mathcal{R}) = \{3\}$, and $(\mathcal{R}') = \{\overline{3}\}$. The form of the gadget in this case is given by a 2×2 matrix of the form

$$\tilde{g}(\mathcal{R},\,\mathcal{R}') = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{4.5}$$

Here the quantities $\mathcal{R}$ and $\mathcal{R}'$ are to be regarded as indices that each take on the values $\{3\}$ and $\{\overline{3}\}$.

This example shows that when $(\mathcal{R}) \neq (\mathcal{R}')$, the gadget can produce real but negative values. Finally, although we will not carry out the calculations, a further informative example consists of working out explicitly the values of the gadget for the case of $(\mathcal{R}) = \{3\}$, and $(\mathcal{R}') = \{8\}$.

When there exists two representations $(\mathcal{R}) \neq (\mathcal{R}')$ of a Lie algebra, where both are represented by $d \times d$ matrices, in addition to forming the traditional d-coefficients, there is another possibility to form a rank four tensor

$$\mathcal{H}^{[(\mathcal{R}),(\mathcal{R}')]}_{A\,B\,C\,D} = \mathrm{Tr}\left[\left\{\, \boldsymbol{t}^{(\mathcal{R})}_A\,,\ \boldsymbol{t}^{(\mathcal{R})}_B \right\}\left\{\, \boldsymbol{t}^{(\mathcal{R}')}_C\,,\ \boldsymbol{t}^{(\mathcal{R}')}_D \right\}\right]. \tag{4.6}$$

In the works of [1–4], the concept of the gadget was extended beyond matrix representations of compact Lie algebras to the realm of adinkra network valise graphs and 0-brane reduced four dimensional minimal SUSY supermultiplets. These works were enabled due to the elucidation of a rank four tensor that exists in these systems which was used to play the role of the d-coefficients. This rank four tensor was given the name of the "holoraumy" tensor and it is analogous to the Lie algebraic tensor defined in (4.6).

5 Observations and summary

To recapitulate, we have shown the existence of a metric[2] over the representation space of minimal off-shell 4D, $\mathcal{N} = 1$ supermultiplets, given by

$$\begin{aligned} \widehat{\mathcal{G}}[(\widehat{\mathcal{R}}),(\widehat{\mathcal{R}}')] = -\frac{1}{768}\Bigg\{ & [\boldsymbol{H}^{\mu(\widehat{\mathcal{R}})}]_{a\,b\,c}{}^{d}\,[\boldsymbol{H}_{\mu}{}^{(\widehat{\mathcal{R}}')}]^{a\,b}{}_{d}{}^{c} \\ & -\frac{1}{2}\,(\gamma^{\alpha})_{c}{}^{e}\,[\boldsymbol{H}^{\mu(\widehat{\mathcal{R}})}]_{a\,b\,e}{}^{f}\,(\gamma_{\alpha})_{f}{}^{d}\,[\boldsymbol{H}_{\mu}{}^{(\widehat{\mathcal{R}}')}]^{a\,b}{}_{d}{}^{c} \\ & -\frac{1}{2}\,(\gamma^{5}\gamma^{\alpha})_{c}{}^{e}\,[\boldsymbol{H}^{\mu(\widehat{\mathcal{R}})}]_{a\,b\,e}{}^{f}\,(\gamma^{5}\gamma_{\alpha})_{f}{}^{d}\,[\boldsymbol{H}_{\mu}{}^{(\widehat{\mathcal{R}}')}]^{a\,b}{}_{d}{}^{c}\Bigg\}\,, \end{aligned} \tag{5.1}$$

[2]This is one member of a class of such metrics.

(where the coefficients are defined by (2.5), (2.6), and (2.7)) has elements identical (over the different supermultiplet representations) to those in the metric

$$\mathcal{G}\left[(\mathcal{R}),\,(\mathcal{R}')\right] = -\frac{1}{48}\sum_{\mathrm{I,J}}\mathrm{Tr}\left[\,\widetilde{V}^{(\mathcal{R})}_{\mathrm{I\,J}}\widetilde{V}^{(\mathcal{R}')}_{\mathrm{I\,J}}\,\right]\,, \tag{5.2}$$

(where these coefficients are defined by (3.4), (3.5), and (3.6)) for the adjacency matrices of three corresponding adinkra networks shown in illustrations 1, 2, and 3.

We have thus, *for the first time and directly in four dimensions*, realized the possibility to define a consistent geometrical viewpoint of the three minimal off-shell $\mathcal{N}=1$ supermultiplets as elements in a representation space with a metric.

We established an equation

$$\widehat{\mathcal{G}}[(\widehat{\mathcal{R}}),(\widehat{\mathcal{R}}')\,] = \mathcal{G}\left[(\mathcal{R}),(\mathcal{R}')\,\right]\,, \tag{5.3}$$

which provides a realization of the concept of “SUSY holography,” (i.e. the proposal that adinkras are holograms of supermultiplets). The existence of such an equation is critical for the entire program we initiated with the work of [5].

The emergence of a four dimensional holoraumy structure among the minimal off-shell $\mathcal{N}=1$ supermultiplets can be seen from the Lorentz representation structure of the three equations in (2.8), (2.9), and (2.10). Due to the appearance of the commutator of the SUSY charges on the right hand side of each equation, one should generally expect the left hand side of the equations to contain the scalar, psuedoscalar, and axial vector Lorentz representations induced by the a and b indices. It is striking that *only* the axial vector occurs for the chiral multiplet fermion, while the calculations for the fermions in the other two supermultiplets contains *all three* expected representations.

We have fixed our normalizations so $\widehat{\mathcal{G}}$ implies the CS, VS, and TS representations correspond to unit vectors in a representation space. The unit vector representing the CS representation is orthogonal to the unit vectors representing the VS, and TS representations. We define an angle between any two of the 4D, $\mathcal{N}=1$ supermultiplet representations $(\widehat{\mathcal{R}})$ and $(\widehat{\mathcal{R}}')$ via the definition

$$cos\left\{\theta[(\widehat{\mathcal{R}})\,,\,(\widehat{\mathcal{R}}')]\right\} = \frac{\widehat{\mathcal{G}}[\,(\widehat{\mathcal{R}}),(\widehat{\mathcal{R}}')\,]}{\sqrt{\widehat{\mathcal{G}}[\,(\widehat{\mathcal{R}}),(\widehat{\mathcal{R}})\,]}\,\sqrt{\widehat{\mathcal{G}}[\,(\widehat{\mathcal{R}}'),(\widehat{\mathcal{R}}')]}}\,. \tag{5.4}$$

The angles between the VS, and TS representations can be read from the matrix given in (2.12) or (3.10) to have the common of θ_{TV} where

$$cos(\theta_{TV}) = -\frac{1}{3}\,. \tag{5.5}$$

On the space of the CS, VS, and TS representations, this appears in figure 4 in agreement with the works of [1, 2].

The process of removing degrees of freedom by dimensional reduction is an example of an ‘injection.” By this we mean there exist many consistent prescriptions for constructing different adinkra shadows for any single higher dimensional supermultiplet. In terms of

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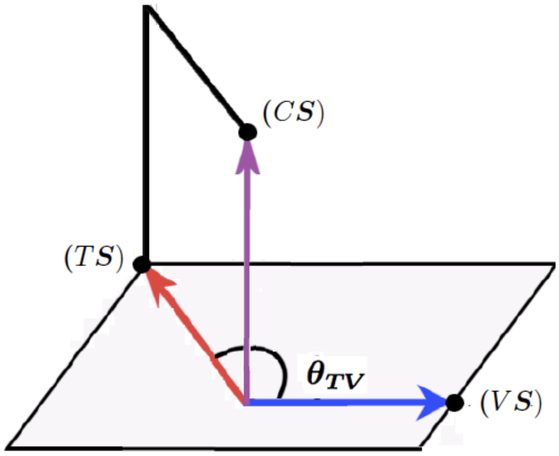

Figure 4. Illustration of the CS-VS-TS subspace using the $\widehat{\mathcal{G}}$ metric.

thinking of the process as a map, there exist many consistent processes that take one supermultiplet and inject it into a "sea" of adinkra networks. Explicit examples of this can now be completely and thoroughly discussed due to the results uncovered in the work of [18].

The key to our progress of this paper is the existence of the four dimensional Lorentz covariant holoraumy tensor dependent upon four integers p, q, r, and s in the following formula

$$
\begin{aligned}
[\boldsymbol{H}^{\mu}(\mathrm{p},\mathrm{q},\mathrm{r},\mathrm{s})]_{abc}{}^{d} = & -i2\,\mathrm{p}\, C_{ab}\,(\gamma^{\mu})_{c}{}^{d} - i2\,\mathrm{q}\,(\gamma^{5})_{ab}(\gamma^{5}\gamma^{\mu})_{c}{}^{d} \\
& - i\,\mathrm{r}\,(\gamma^{5}\gamma^{\nu})_{ab}\,(\gamma^{5}\,[\gamma_{\nu}\,,\,\gamma^{\mu}])_{c}{}^{d} \;-\; i\,2\,\mathrm{s}\,(\gamma^{5}\gamma^{\mu})_{ab}\,(\gamma^{5})_{c}{}^{d}\,,
\end{aligned}
\tag{5.6}
$$

and corresponding to the three vectors illustrated above we have

Supermultiplet	p	q	r	s
CS	0	0	$+1$	0
VS	$+1$	$+1$	0	$+1$
TS	-1	$+1$	0	-1

that encodes the distinct supermultiplets in an extremely compact manner.

This method of classifying the minimal off-shell supermultiplets makes a finer distinction than we gave in the works of [12] or [16]. In the first of these, a quantum number χ_{o} ("Kye-Oh") was introduced initially via adinkra networks and extended to supermultiplets in the second. Comparing these previous works to the formula in (5.6), we see

$$
\chi_{\mathrm{o}} = (-1)^{\mathrm{s}^2}\,, \tag{5.7}
$$

with regard to the minimal off-shell supermultiplet representations.

If we include the parity reflected and variant representation versions of the minimal (see [4]) supermultiplets there are only eight such representations. It is thus possible to extend the minimal supermultiplet representation label $\widehat{\mathcal{R}}$ that appears on the left hand side of (5.3) to cover an eight dimensional vector space. On the other hand the work of [18] implies the existence of 1,536 sets of consistent adinkra networks based on the existence of a Coxeter group. This means it is possible to extend the minimal adinkra network representation label $\mathcal{R}$ that appears on the right hand side of (5.3) to cover a 1,536 dimensional vector space!

We are now in position to state a conjecture about 4D, $\mathcal{N} = 1$ minimal off-shell supermultiplets and 1d, $N = 4$ minimal adinkras constructed from a Coxeter group. Let $\widehat{\mathcal{R}}_1, \ldots, \widehat{\mathcal{R}}_8$ denote any of the minimal off-shell 4D, $\mathcal{N} = 1$ supermultiplets. Let $\mathcal{R}_1, \ldots, \mathcal{R}_{1,536}$ denote any of the minimal four color adinkra based on the Coxeter group described in [18]. Let $\mathcal{R}^*_1, \cdots, \mathcal{R}^*_8$ denote any eight among the 1,536 adinkra representations.

We conjecture that if the equation in (5.3) is satisfied for some ordering of the supermultiplets $\widehat{\mathcal{R}}_1, \ldots, \widehat{\mathcal{R}}_8$ together with an appropriate ordering of adinkra network representations $\mathcal{R}^*_1, \cdots, \mathcal{R}^*_8$, then there must exist a projection operator $\mathcal{P}$ such that

$$\begin{aligned} \mathcal{P} : \widehat{\mathcal{R}}_1 &\to \mathcal{R}^*_1\,, \\ \mathcal{P} : \widehat{\mathcal{R}}_2 &\to \mathcal{R}^*_2\,, \\ &\vdots \\ \mathcal{P} : \widehat{\mathcal{R}}_8 &\to \mathcal{R}^*_8\,. \end{aligned} \tag{5.8}$$

An important implication of this conjecture is that the illustrations 1, 2, and 3 provide only one consistent set of adinkra representations of the minimal off-shell 4D, $\mathcal{N} = \infty$ supermultiplets. Any three adinkra networks that preserve the conditions in (5.3) can act as the shadows for the four dimensional supermultiplets.

There can easily arise a problem associated with starting from a chosen set of adinkra networks along with the data they contain and attempting to reconstruct the higher dimensional supermultiplets to be associated with these chosen networks. Only sets of adinkras that satisfy the equation in (5.3) should be identified with the higher dimensional supermultiplets. Stated another way, the condition in (5.3) acts as a filter for how one begins from adinkras and then uses these as a tool to reconstruct higher dimensional supermultiplets.

We believe ultimately, the condition in (5.3) will play a very important role in the program that has been initiated by the work of [19] which has as its aim to place the representation theory of supersymmetry into the context of algebraic geometry and Riemann surfaces.

In this work, we have built upon the foundation that has been provided by careful and detailed studies of the case of four dimensional $\mathcal{N} = \infty$ off-shell minimal supermultiplets and minimal four-color adinkras. There is still more work to be done in order to extend the formula in (5.6) for the Lorentz covariant holoraumy tensor to cover other four dimensional $\mathcal{N} = \infty$ off-shell supermultiplets... and beyond. It is our expectation that for more complication supermultiplets, the 4D holoraumy tensors we have found will likely have to by augmented by additional ones, perhaps with a different Lorentz representation structure. However, with the success demonstrated in this current work, we are confident about the success of such efforts to be undertaken in the future.

All truths are easy to understand once they are discovered;
the point is to discover them.

– Galileo Galilei

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A Components of the 4D gadget calculations

In the appendix, we provide some of the intermediate calculations that lead to the results in (2.13). We begin with the 4D holoraumy tensors defined by Next, we need to perform a series of "conjugation" transformations on each of these by acting with the matrices (γ^α), $([\gamma^\alpha\,,\ \gamma^\beta])$, $(\gamma^5\gamma^\alpha)$, and (γ^5) respectively.

$$
\begin{aligned}
(\gamma^\alpha)_c{}^e \left[\boldsymbol{H}^{\mu(CS)}\right]_{a\,b\,e}{}^f\,(\gamma_\alpha)_f{}^d &= 0\,,\\
([\,\gamma^\alpha\,,\ \gamma^\beta\,]))_c{}^e \left[\boldsymbol{H}^{\mu(CS)}\right]_{a\,b\,e}{}^f\,([\,\gamma_\alpha\,,\ \gamma_\beta\,]))_f{}^d &= 16\,\left[\boldsymbol{H}^{\mu(CS)}\right]_{a\,b\,c}{}^d\,,\\
(\gamma^5\gamma^\alpha)_c{}^e \left[\boldsymbol{H}^{\mu(CS)}\right]_{a\,b\,e}{}^f\,(\gamma^5\gamma_\alpha)_f{}^d &= 0\,,\\
(\gamma^5)_c{}^e \left[\boldsymbol{H}^{\mu(CS)}\right]_{a\,b\,e}{}^f\,(\gamma^5)_f{}^d &= \left[\boldsymbol{H}^{\mu(CS)}\right]_{a\,b\,c}{}^d\,,
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
(\gamma^\alpha)_c{}^e \left[\boldsymbol{H}^{\mu(VS)}\right]_{a\,b\,e}{}^f\,(\gamma_\alpha)_f{}^d &= +i\,4\,C_{ab}(\gamma^\mu)_c{}^d - i\,4\,(\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d\\
&\quad + i\,8\,(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d\,,\\
([\,\gamma^\alpha\,,\ \gamma^\beta\,]))_c{}^e \left[\boldsymbol{H}^{\mu(VS)}\right]_{a\,b\,e}{}^f\,([\,\gamma_\alpha\,,\ \gamma_\beta\,]))_f{}^d &= i\,96\,(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d\,,\\
(\gamma^5\gamma^\alpha)_c{}^e \left[\boldsymbol{H}^{\mu(VS)}\right]_{a\,b\,e}{}^f\,(\gamma^5\gamma_\alpha)_f{}^d &= i4\,C_{ab}(\gamma^\mu)_c{}^d - i\,4\,(\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d\\
&\quad - i8\,(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d\,,\\
(\gamma^5)_c{}^e \left[\boldsymbol{H}^{\mu(VS)}\right]_{a\,b\,e}{}^f\,(\gamma^5)_f{}^d &= i\,2\,C_{ab}(\gamma^\mu)_c{}^d + i\,2\,(\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d\\
&\quad - i\,2\,(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d\,,
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
(\gamma^\alpha)_c{}^e \left[\boldsymbol{H}^{\mu(TS)}\right]_{a\,b\,e}{}^f\,(\gamma_\alpha)_f{}^d &= -i\,4\,C_{ab}(\gamma^\mu)_c{}^d - i\,4\,(\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d\\
&\quad - i\,8\,(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d\,,\\
([\,\gamma^\alpha\,,\ \gamma^\beta\,]))_c{}^e \left[\boldsymbol{H}^{\mu(TS)}\right]_{a\,b\,e}{}^f\,([\,\gamma_\alpha\,,\ \gamma_\beta\,]))_f{}^d &= -\,i\,96\,(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d\,,\\
(\gamma^5\gamma^\alpha)_c{}^e \left[\boldsymbol{H}^{\mu(TS)}\right]_{a\,b\,e}{}^f\,(\gamma^5\gamma_\alpha)_f{}^d &= -i\,4\,C_{ab}(\gamma^\mu)_c{}^d - i\,4\,(\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d\\
&\quad + i\,8\,(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d\,,\\
(\gamma^5)_c{}^e \left[\boldsymbol{H}^{\mu(TS)}\right]_{a\,b\,e}{}^f\,(\gamma^5)_f{}^d &= -i\,2\,C_{ab}(\gamma^\mu)_c{}^d + i\,2\,(\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d\\
&\quad + i\,2\,(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d\,.
\end{aligned}
\tag{A.3}
$$

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